Journal of Algebra 323 (2010) 658–670

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On Néron models of moduli spaces of theta characteristics

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ARTICLE INFO

Article history:
Received 17 July 2008
Available online 13 November 2009
Communicated by Steven Dale Cutkosky

Keywords:
Jacobian of a curve
Spin curve
Néron model

ABSTRACT

Let $f:\mathcal{C}\to B$ be a smoothing of a stable curve C and S_f^* be the moduli space of theta characteristics on the smooth fibers of f. We describe the Néron model $N(S_f^*)$, in terms of combinatorial invariants of the dual graph of C. Furthermore, we provide a modular description of $N(S_f^*)$ and we construct an immersion $\psi_f : N(S_f^*) \hookrightarrow J_{\mathcal{E}}^{\sigma}$, where $J_{\mathcal{E}}^{\sigma}$ is a suitable relative compactified Jacobian. We show that ψ_f factors through the locus of $J_{\mathcal{E}}^{\sigma}$ parametrizing locally free rank-1 sheaves.

1. Introduction

1.1. Étale models

Let C be a projective scheme of dimension 1 over an algebraically closed field of characteristic zero, or, for short, *a curve*. Let $f:\mathcal{C}\to B$ be a *general smoothing* of C, i.e. a family of curves over a smooth and connected curve B, where $\mathcal{C}$ is non-singular and where $C=f^{-1}(0)$ for some $0\in B$ and $\mathcal{C}^*=f^{-1}(B-0)$ is smooth over $B-0$. Let S_f^* be the moduli scheme of theta characteristics on the fibers of $f|_{\mathcal{C}^*}$, an étale scheme over $B-0$. It makes sense to ask: is it possible to give a description of the maximal étale B-model of S_f^*, via combinatorial invariants of C? A goal of this paper is to give a positive answer to this question, when C is a stable curve without non-trivial automorphisms.

This distinguished B-model is necessarily the Néron model $N(S_f^*)$ of S_f^* over B. More generally, the Néron model provides a smooth and separated B-model of a scheme defined over the field of fractions of B. The Néron model is canonically determined by a universal property, known as *the Néron mapping property*. Recall that the theory of Néron models has been introduced in [N] for abelian

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[1] Supported by CNPq (Proc. 151610/2005-3) and by Faperj (Proc. E-26/152-629/2005).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2009.11.005

varieties and it became apparent in [R] their connection with the Picard functor. They have been employed in arithmetic and geometry and recently also in the moduli theory of curves (see [C2,Ch,B]).

The posed question has been recently considered in [Ch]. There, it is shown a necessary and sufficient condition for the existence of a finite Néron model of the moduli space of r-torsion line bundles on the fibers of $f|_{\mathcal{C}^*}$, via combinatorial invariants of the semistable reduction of $\mathcal{C}$. More generally, one can consider the Néron model $N(\operatorname{Pic}^d \mathcal{C}^*)$, where $\operatorname{Pic}^d \mathcal{C}^*$ is the degree-d relative Jacobian of $f|_{\mathcal{C}^*}$. Assume that C is a stable curve of genus g and d an integer such that $(d-g+1, 2g-2) = 1$. Let $\overline{P}_{d,g}$ be the universal Picard variety over $\overline{M}_g$ constructed in [C1]. In [C2, Theorem 6.1], it is shown that $N(\operatorname{Pic}^d \mathcal{C}^*) = B \times_{\overline{\mathcal{M}}_g} \mathcal{P}_{d,g}$, where $\mathcal{P}_{d,g}$ is the representable stack version of the open subset of $\overline{P}_{d,g}$ parametrizing equivalence classes of balanced line bundles on stable curves. However, a theta characteristic of a curve of genus g has degree $g-1$, then [C2, Theorem 6.1] does not hold in this case. To find a geometric description of $N(S_f^*)$, we will need to consider different compactified Jacobians.

1.2. The main result

Fix a smoothing $f : \mathcal{C} \to B$ of a stable curve C and let $d = g-1$. There are plenty of degree-d relative compactified Jacobian. In [A], it is shown that the ones constructed in [C1,OS,S] are all isomorphic. A different degree-d relative compactified Jacobian was constructed in [AK] for a family of integral curves and more generally in [E] for a family of reduced and connected curves. This compactified Jacobian is denoted by $J_{\mathcal{E}}^{\sigma}$. We establish a relationship between $J_{\mathcal{E}}^{\sigma}$ and the Néron model $N(S_f^*)$. A comparison result between $J_{\mathcal{E}}^{\sigma}$ and Néron models of Picard schemes is contained in [B]. However, the fact that we are considering the subfunctor of theta characteristics, allows us to find a rather explicit geometric description of $N(S_f^*)$.

Since we work over an algebraically closed field of characteristic zero, we can apply the techniques and results of [CCC]. There, for a given line bundle $\mathcal{G}$ of $\mathcal{C}$, the authors constructed a scheme $\overline{S_f}(\mathcal{G})$, finite over B, compactifying the moduli space of pairs (C_b, L_b), C_b a fiber of f and L_b a square root $\mathcal{G}|_{C_b}$. The objects employed in this construction are *limit square roots*, that are certain line bundles supported on nodal curves obtained by blowing-up the curves of the family, i.e. curves obtained by replacing nodal singularities by rational curves, called *exceptional curves*. There is a distinguished combinatorial invariant attached to a blow-up X of a curve, which is the graph Σ_X whose vertices are the connected components of the residual in X of the union of the exceptional components and whose edges are the exceptional components of X. We will describe $N(S_f^*)$ via combinatorial properties of the graph Σ_X, as follows.

In Section 3.2 we introduce and classify the set $\operatorname{Ad}_f(C_0)$ of *f-admissible twisters of C with respect to C_0*, where C_0 is an irreducible component of C. The set $\operatorname{Ad}_f(C_0)$ is a subset of the set of line bundles on C that are limits of trivial bundles on the smooth fibers of f. Our main result, contained in Theorems 2.2 and 3.9, is:

Theorem 1.1. *Let $f : \mathcal{C} \to B$ be a general smoothing of a stable curve C of genus $g \geqslant 3$ with $\operatorname{Aut}(C) = \{id\}$. Let $\nu : \overline{S_f^{\nu}}(\omega_f) \to \overline{S_f}(\omega_f)$ be the normalization map. Then the following properties are equivalent for every $\xi \in \overline{S_f}(\omega_f)$:*

(i) *$\overline{S_f^{\nu}}(\omega_f) \to B$ is étale at a point of $\nu^{-1}(\xi)$;*
(ii) *$\overline{S_f^{\nu}}(\omega_f) \to B$ is étale at any point of $\nu^{-1}(\xi)$;*
(iii) *if ξ is supported on the blow-up X of C, then Σ_X is bipartite.*

Furthermore, for every irreducible component $C_0 \subset C$ we have:

$$N(S_f^*) \simeq \frac{\bigcup_{T \in \operatorname{Ad}_f(C_0)} S_f(\omega_f \otimes T)}{\sim}$$

where $S_f(\omega_f \otimes T)$ is the open subscheme of $\overline{S_f}(\omega_f \otimes T)$ parametrizing limit square roots supported on stable curves and where $\sim$ denotes the gluing along the generic fiber of $S_f(\omega_f \otimes T) \to B$.

Let $\mathcal{E}$ be the polarization $\mathcal{E} = \mathcal{O}_{\mathcal{C}}$ and let $(J^\sigma_{\mathcal{E}})^{free}$ be the open subspace of $J^\sigma_{\mathcal{E}}$ parametrizing locally free sheaves. Then there exists an immersion:

$$\psi_f : N(S^*_f) \hookrightarrow (J^\sigma_{\mathcal{E}})^{free}.$$

The idea of comparing moduli spaces of roots of line bundles and compactified Jacobians already appears in [CCC] and [F]. Using Theorem 1.1, we are able to recover the combinatorial result of [Ch], which classifies the curves for which $N(S^*_f)$ is finite over B in term of their dual graph (see Proposition 2.4).

1.3. Notation and terminology

A *curve* is a connected, projective, reduced scheme of dimension 1 over an algebraically closed field of characteristic zero. A *stable* (*semistable*) curve C is a nodal curve such that every smooth rational component meets the rest of the curve in at least 3 points (2 points). The genus of C is $g_C = h^0(C, \omega_C)$, where ω_C is the dualizing sheaf of C. If $Z \subset C$ is a subcurve, the *residual in C of Z* is $Z^c := \overline{C - Z}$.

A *family of curves* is a proper and flat morphism $f : \mathcal{C} \to B$ whose fibers are curves. If $b \in B$, we denote by $C_b = f^{-1}(b)$. A *smoothing* of a curve C is a family $f : \mathcal{C} \to B$, where B is a smooth, connected, affine curve with a distinguished point $0 \in B$ such that $\mathcal{C}^* := f^{-1}(B - 0)$ is smooth over $B - 0$ and $C = f^{-1}(0)$. A *general smoothing* is a smoothing with $\mathcal{C}$ smooth.

A nodal curve X is *obtained by blowing-up* a nodal curve C at a subset Δ of nodes of C, if there is a morphism $\pi : X \to C$ such that, for every $p_i \in \Delta$, $\pi^{-1}(p_i) = E_i \simeq \mathbb{P}^1$ and $\pi : X - \bigcup_i E_i \to C - \Delta$ is an isomorphism. For every $p_i \in \Delta$, we call E_i an *exceptional component*. A family of curves $\mathcal{X} \to B$ is a *family of blow-ups of a family $\mathcal{C} \to B$* if there exists a B-morphism $\pi : \mathcal{X} \to \mathcal{C}$ such that $\pi|_{X_b} : X_b \to C_b$ is obtained by blowing-up C_b, for every $b \in B$.

Let I be a coherent sheaf on a curve C. We say that I is *torsion-free* if its associated points are generic points of C. We say that I is of *rank* 1 if I is invertible on a dense open subset of C. We say that I is *simple* if $\mathrm{End}(I) = k$. Each line bundle on C is torsion-free of rank 1 and simple. If I is torsion-free of rank 1, we call $\deg(I) := \chi(I) - \chi(\mathcal{O}_C)$ the *degree* of I.

Denote by $\mathrm{Aut}(C)$ the group of automorphism of a curve C. If Γ is a graph with an orientation, then $\delta : \mathcal{C}^0(\Gamma, \mathbb{Z}/2\mathbb{Z}) \to \mathcal{C}^1(\mathbb{Z}/2\mathbb{Z})$ denotes the coboundary operator. A graph Γ is *bipartite* if there is a partition of its vertices into two sets A and B such that each edge of Γ has a vertex in A and the other vertex in B. Equivalently, Γ is bipartite if each cycle of Γ has an even number of edges.

2. Néron models of moduli spaces of square roots

2.1. Review of moduli spaces of limit square roots

Let C be a nodal curve and $G \in \mathrm{Pic}(C)$ of even degree. Consider a tern (X, L, α), where $\pi : X \to C$ is a blow-up of C, $L \in \mathrm{Pic}\, X$ and α is a homomorphism $\alpha : L^{\otimes 2} \to \pi^*(G)$. Then (X, L, α) is a *limit square root* of (C, G) if:

(i) the restriction of L to every exceptional component has degree 1;
(ii) the map α is an isomorphism at the points of X not belonging to an exceptional component;
(iii) for every exceptional component E such that $E \cap E^c = \{p, q\}$ the orders of vanishing of α at p and q add up to 2.

The curve X is called the *support* of the limit square root. If $\mathcal{C} \to B$ is a family of stable curves and $\mathcal{G} \in \mathrm{Pic}\,\mathcal{C}$ has even relative degree, then a *limit square root* of $(\mathcal{C}, \mathcal{G})$ is a tern $(\mathcal{X}, \mathcal{L}, \alpha)$, where

$\pi:\mathcal{X}\to\mathcal{C}$ is a family of blow-ups, $\mathcal{L}\in\operatorname{Pic}\mathcal{X}$ and α is a homomorphism $\alpha:\mathcal{L}^{\otimes 2}\to\pi^*\mathcal{G}$ such that $(X_b,\mathcal{L}|_{X_b},\alpha|_{X_b})$ is a limit square root of $(C_b,\mathcal{G}|_{C_b})$, for every $b\in B$.

If X is obtained by blowing-up the curve C, set $\widetilde{X}:=\overline{X-\bigcup_{E\in\mathcal{E}(X)}E}$, where $\mathcal{E}(X)$ is the set of exceptional components of X.

Remark 2.1. There exists a notion of isomorphism of limit square roots. By [Co, Lemma 2.1], two limit square roots (X,L,α) and (X,L',α') are isomorphic if and only if $L|_{\widetilde{X}}\simeq L'|_{\widetilde{X}}$.

Let $f:\mathcal{C}\to B$ be a family of nodal curves over a quasi-projective scheme B and $\mathcal{G}\in\operatorname{Pic}(\mathcal{C})$ of even relative degree. Let $\overline{\boldsymbol{\mathcal{S}}_f}(\mathcal{G})$ be the contravariant functor from the category of locally Noetherian B-schemes to sets, defined on T by:

$$\overline{\boldsymbol{\mathcal{S}}_f}(\mathcal{G})(T):=\{\text{limit square roots of } q^*\mathcal{G}\}/\sim \tag{2.1}$$

where $q:\mathcal{C}\times_B T\to\mathcal{C}$ is the first projection and $\sim$ means isomorphism of limit square roots. There exists a quasi-projective scheme $\overline{S_f}(\mathcal{G})$, finite over B, which coarsely represents $\overline{\boldsymbol{\mathcal{S}}_f}(\mathcal{G})$. For more details, we refer to [CCC, Theorem 2.4.1]. Abusing notation, we will often denote by ξ both the isomorphism class of a limit square root and the point of $\overline{S_f}(\mathcal{G})$ parametrizing this equivalence class.

Let C be a nodal curve and $G\in\operatorname{Pic}(C)$ of even degree. Denote by $\overline{S_C}(G)$ the zero-dimensional scheme $\overline{S_{f_C}}(G)$, where $f_C:C\to\operatorname{Spec}(k)$ is the structure morphism of C. In particular, $\overline{S_C}(G)$ is in bijection with the isomorphism classes of limit square roots of (C,G). If $f:\mathcal{C}\to B$ is a family of curves and $\mathcal{G}\in\operatorname{Pic}\mathcal{C}$, then the fiber of $\overline{S_f}(\mathcal{G})\to B$ over $b\in B$ is $\overline{S_{C_b}}(\mathcal{G}|_{C_b})$. If $f:\mathcal{C}\to B$ is a smoothing of a stable curve C with distinguished point $0\in B$ and $\mathcal{G}$ is a line bundle on $\mathcal{C}$ of even relative degree, let $\mathcal{C}^*:=f^{-1}(B-0)$ and $\mathcal{G}^*=\mathcal{G}|_{\mathcal{C}^*}$ and denote $S(\mathcal{G}^*):=\overline{S_{f|_{\mathcal{C}^*}}}(\mathcal{G}^*)$. Moreover, denote by $S_f(\mathcal{G})$ the open subscheme of $\overline{S_f}(\mathcal{G})$ parametrizing limit square roots supported on stable curves.

Let X be obtained by blowing-up C. Let Σ_X be the graph whose vertices (resp. edges) corresponds to the connected components of $\widetilde{X}$ (resp. to the exceptional component of X), where an edge connects two vertices if the corresponding exceptional component intersects the corresponding connected components. By [CCC, 4.1], the multiplicity of $\overline{S_C}(G)$ in ξ is $2^{b_1(\Sigma_X)}$, if (X,L,α) is a representative of ξ. If C is a stable curve, denote by Γ_C the usual dual graph of C, whose edges (resp. vertices) corresponds to the nodes (resp. to the irreducible components) of C. Let Γ_X the subgraph of Γ_C whose edges corresponds to the nodes of C which are not blown-up to get X. As observed in [CCC], the graph Σ_X is obtained from Γ_C by contracting the edges contained in Γ_X.

2.2. A combinatorial result on the Néron model of $S(\mathcal{G}^*)$

Let B be a connected Dedekind scheme with field of fractions K. Let X_K be a smooth and separated K-scheme of finite type. A Néron model of X_K is a B-scheme $N(X_K)$, which is a smooth, separated and finite type B-model of X_K and satisfying the following universal property, well-known as *Néron mapping property*: for every smooth B-scheme Y and K-morphism $\phi_K:Y_K\to X_K$, there exists a unique extension of ϕ_K to a B-morphism $\phi:Y\to N(X_K)$. If a Néron model exists, it is canonically determined, up to a unique isomorphism, by the Néron mapping property.

Theorem 2.2. *Let $f:\mathcal{C}\to B$ be a general smoothing of a stable curve C of genus $g\geqslant 3$ with* $\operatorname{Aut}(C)=\{id\}$. *Consider the moduli space $\overline{S_f}(\mathcal{G})$, where $\mathcal{G}\in\operatorname{Pic}\mathcal{C}$ is of even relative degree, and its normalization $\nu:\overline{S^{\nu}_f}(\mathcal{G})\to\overline{S_f}(\mathcal{G})$. Then the Néron model of $S(\mathcal{G}^*)$ is isomorphic to the étale locus of $\overline{S^{\nu}_f}(\mathcal{G})\to B$ and the following properties are equivalent for every $\xi\in\overline{S_f}(\mathcal{G})$:*

(i) *$\overline{S^{\nu}_f}(\mathcal{G})\to B$ is étale at a point of $\nu^{-1}(\xi)$;*
(ii) *$\overline{S^{\nu}_f}(\mathcal{G})\to B$ is étale at any point of $\nu^{-1}(\xi)$;*
(iii) *if X is the support of a representative of ξ, then Σ_X is bipartite.*

Proof. Let $\gamma_f : B \to \overline{M_g}$ be the moduli morphism, where $\overline{M_g}$ is the moduli space of Deligne–Mumford stable curves. Since $\mathcal{C}$ is smooth and $\mathrm{Aut}(C) = \{id\}$, the image of γ_f is smooth at $\gamma_f(0)$. Up to shrink B to an open (analytic) subset containing 0, we can assume $B \subset \mathrm{Def}(C)$, where $\mathrm{Def}(C)$ is the base of the universal deformation of C. Let (X, L, α) be a representative of some $\xi \in \overline{S_C}(\mathcal{G}|_C)$. Assume that X is obtained by blowing-up the nodes $n_1, \dots, n_m$ of C. Let t_j be the coordinate of $\mathrm{Def}(C)$ such that $\{t_j = 0\}$ is the locus where the node n_j persists, for every $j = 1, \dots, m$. Using the fact that $\mathcal{C}$ is smooth and the implicit function theorem, we can describe B as:

$$\big(t_1, t_1 h_2(t_1), t_1 h_3(t_1), \dots, t_1 h_{3g-3}(t_1)\big)$$

where h_i is an analytic function such that $h_i(0) \in \mathbb{C}^*$, for $i = 2, \dots, m$. Consider the morphism $\rho : \mathrm{Def}(C) \to \mathrm{Def}(C)$ given by:

$$(t_1, \dots, t_m, t_{m+1}, \dots, t_{3g-3}) \xrightarrow{\rho} \big(t_1^2, \dots, t_m^2, t_{m+1}, \dots, t_{3g-3}\big).$$

Pick $U_\xi = \rho^{-1}(B)$. Fix an orientation on the graph Σ_X and let $e_1, \dots, e_m$ be the edges of Σ_X, corresponding to the exceptional components of X. Consider the coboundary operator $\delta : \mathcal{C}^0(\Sigma_X, \mathbb{Z}/2\mathbb{Z}) \to \mathcal{C}^1(\Sigma_X, \mathbb{Z}/2\mathbb{Z})$. By [CCC, Lemmas 2.3.2 and 3.3.1], the moduli space $\overline{S_f}(\mathcal{G})$ is $U_\xi / \mathrm{Im}(\delta)$, locally analytically at ξ. Here, an element $\theta = \sum_{i=1}^m \epsilon_i \cdot e_i \in \mathcal{C}^1(\Sigma_X, \mathbb{Z}/2\mathbb{Z})$, where $\epsilon_i \in \mathbb{Z}/2\mathbb{Z}$ for $i = 1, \dots, m$, acts on U_ξ via:

$$(t_1, \dots, t_m, t_{m+1}, \dots, t_{3g-3}) \xrightarrow{\theta} (\epsilon_1 t_1, \dots, \epsilon_m t_m, t_{m+1}, \dots, t_{3g-3}).$$

Furthermore, $\rho|_{U_\xi}$ factors through a morphism $\mu : U_\xi / \mathrm{Im}(\delta) \to B$, giving locally the finite morphism $\overline{S_f}(\mathcal{G}) \to B$ described in Section 2.1.

The tangent cone of U_ξ at the origin is:

$$T_0(U_\xi) = \big\{t_2^2 - h_2(0)t_1^2 = 0,\ \dots,\ t_m^2 - h_m(0)t_1^2 = 0,\ t_{m+1} = 0,\ \dots,\ t_{3g-3} = 0\big\}.$$

Hence U_ξ has 2^{m-1} distinct branches intersecting transversally. Consider the automorphisms θ^- of $\mathrm{Def}(C)$ defined as:

$$(t_1, \dots, t_m, t_{m+1}, \dots, t_{3g-3}) \xrightarrow{\theta^-} (-t_1, \dots, -t_m, t_{m+1}, \dots, t_{3g-3}).$$

Notice that θ^- commutes with ρ and acts over U_ξ preserving the irreducible components of $T_0(U_\xi)$ and hence also the branches of U_ξ. We see that $\rho|_{U_\xi}$ is a cover of B of degree 2^m and, for every branch $U'_\xi \subset U_\xi$, we have that $\rho|_{U'_\xi}$ is a degree-2 cover of B with involution $\theta^-|_{U'_\xi}$.

Notice that $\theta^- \in \mathrm{Im}(\delta)$ if and only if Σ_X is bipartite.

We show (i) $\Rightarrow$ (iii). Assume that $\overline{S_f^\nu}(\mathcal{G}) \to B$ is étale at a point of $\nu^{-1}(\xi)$. Consider the finite morphism $\mu : U_\xi / \mathrm{Im}(\delta) \to B$, giving locally the morphism $\overline{S_f}(\mathcal{G}) \to B$. Then, for at least one branch $U'_\xi \subset U_\xi$, the restriction $\mu|_{U'_\xi / \mathrm{Im}(\delta)}$ is a bijection. Hence $\theta^-|_{U'_\xi} = \bar{\theta}|_{U'_\xi}$, for some $\bar{\theta} \in \mathrm{Im}(\delta)$, otherwise $\mu|_{U'_\xi / \mathrm{Im}(\delta)}$ would have degree 2. Since θ^- is the only non-trivial automorphism of $\mathcal{C}^1(\Sigma_X, \mathbb{Z}/2\mathbb{Z})$ preserving U'_ξ, then $\theta^- = \bar{\theta} \in \mathrm{Im}(\delta)$ and hence Σ_X is bipartite.

We show (iii) $\Rightarrow$ (ii). If Σ_X is bipartite, then $\theta^- \in \mathrm{Im}(\delta)$ and hence $\mu|_{U'_\xi / \mathrm{Im}(\delta)}$ is a bijection, for every branch $U'_\xi \subset U_\xi$. In particular, $\overline{S_f^\nu}(\mathcal{G}) \to B$ is étale at every points of $\nu^{-1}(\xi)$. The implication (ii) $\Rightarrow$ (i) is trivial.

To prove the first statement, by [BLR, Proposition 1.2.4] we can assume without loss of generality that $B = \mathrm{Spec}\, R$, where R is a discrete valuation ring. By [BLR, Corollary 6.5.4], the Néron model

$N(S(\mathcal{G}^*))$ of $S(\mathcal{G}^*)$ exists. Let $\overline{S_f^\nu}(\mathcal{G})^{et}$ be the étale locus of $\overline{S_f^\nu}(\mathcal{G}) \to B$. Now, $N(S(\mathcal{G}^*))$ is étale over B and it is a birational model of $\overline{S_f}(\mathcal{G})$. Then we have an immersion $N(S(\mathcal{G}^*)) \hookrightarrow \overline{S_f^\nu}(\mathcal{G})^{et}$ and, by the Néron mapping property, a reverse immersion holds as well. $\square$

Lemma 2.3. *Let C be a stable curve and Γ_C its dual graph. Let X be a blow-up of C and Γ_X the subgraph of Γ_C associated to Γ_X as explained in Section* 2.1. *Then X is the support of a representative of some $\xi \in \overline{S_C}(\omega_C)$ if and only if Γ_X can be written as a possibly empty union of cycles of Γ_C whose mutual intersections contains no edge of Γ_C.*

Proof. See [CC, Section 1.3, p. 6]. $\square$

Proposition 2.4. *Let $f:\mathcal{C} \to B$ be a general smoothing of a stable curve C of genus $g \geqslant 3$ with* $\mathrm{Aut}(C) = \{id\}$. *Then $N(S(\omega_f^*))$ is finite over B if and only if for every pair (Γ_1, Γ_2) of cycles of Γ_C the intersection $\Gamma_1 \cap \Gamma_2$ contains an even number of edges of Γ_C.*

Proof. Assume that the condition of the statement holds. If Γ is a cycle of Γ_C, then, applying the condition of the statement to the pair (Γ, Γ), we see that Γ has an even number of edges. In particular Γ_C is bipartite. Pick $\xi \in \overline{S_C}(\omega_C)$ and let X be the support of any representative of ξ. If X is obtained by blowing-up X at the whole set of its nodes, then $\Sigma_X = \Gamma_C$ and Σ_X is bipartite. Otherwise, Σ_X is obtained by contracting Γ_X. Combining Lemma 2.3 and the condition of the statement, we have that the cycles of Σ_X have an even number of edges, and then Σ_X is bipartite. Then Σ_X is bipartite in any case. By Theorem 2.2 we have that $N(S(\omega_f^*)) \simeq \overline{S_f^\nu}(\omega_f)$, then $N(S(\omega_f^*))$ is finite over B.

Conversely, assume that the condition of the statement does not hold. By Theorem 2.2, it suffices to show that $\overline{S_f^\nu}$ is not étale over B. We have two cases. In the first case, there exists a cycle of Γ_C with an odd number of edges, i.e. Γ_C is not bipartite. By Lemma 2.3, there exists a $\xi \in \overline{S_C}(\omega_C)$ with a representative supported on the curve X obtained by blowing-up C at the whole set of its nodes and $\Sigma_X = \Gamma_C$. By Theorem 2.2, $\overline{S_f^\nu}$ is not étale at ξ. In the second case, Γ_C is bipartite and there are two different cycles Γ_1 and Γ_2 such that $\Gamma_1 \cap \Gamma_2$ is an odd number of edges of Γ_C. Consider the graph Σ obtained by Γ_C by contracting Γ_C at the edges of Γ_1. In Σ, the cycle obtained by contracting Γ_2 at the edges of Γ_1 has an odd number of edges, then Σ is not bipartite. Let X_{Γ_1} be obtained by blowing-up C at the nodes whose corresponding edges are not contained in Γ_1. By Lemma 2.3 there is a $\xi \in \overline{S_C}(\omega_C)$ with a representative supported on X_{Γ_1} and $\Sigma_{X_{\Gamma_1}} = \Sigma$. Hence $\Sigma_{X_{\Gamma_1}}$ is not bipartite and $\overline{S_f^\nu}$ is not étale at ξ. $\square$

3. Néron models of $S(\omega_f^*)$ within $J_{\mathcal{E}}^{\sigma}$

3.1. The compactified Jacobian $J_{\mathcal{E}}^{\sigma}$

Let $f:\mathcal{C} \to B$ be a family of curves. Then f *admits enough sections through the B-smooth locus of $\mathcal{C}$* if there are sections $\sigma_1, \dots, \sigma_n : B \to \mathcal{C}$ of f such that:

(i) σ_i factors through the B-smooth locus of $\mathcal{C}$ for $i = 1, \dots, n$;
(ii) for every $b \in B$, every irreducible component of C_b contains $\sigma_i(b)$ for some $i = 1, \dots, n$.

Let $f:\mathcal{C} \to B$ be a family of curves, where B is a locally Noetherian scheme. Assume that f admits enough sections through the B-smooth locus of $\mathcal{C}$. Let $\mathbf{J}_d$ be the contravariant functor from the category of locally Noetherian B-schemes to sets, associating to T the set of equivalence classes of B-flat, coherent sheaves $\mathcal{I}$ on $\mathcal{C} \times_B T/T$ whose fibers over B are degree d, simple, rank-1, torsion-free sheaves. Here, $\mathcal{I}_1$ and $\mathcal{I}_2$ are equivalent if there is an invertible sheaf M on T such that $\mathcal{I}_1 \simeq \mathcal{I}_2 \otimes p^*M$, for $p:\mathcal{C} \times_B T \to T$ the projection. In [E], it is shown that $\mathbf{J}_d$ is finely represented by a scheme J_d. Furthermore, one can consider distinguished subschemes of J_d as follows. Fix an integer d.

A *polarization on* $\mathcal{C}$ is a vector bundle $\mathcal{E}$ on $\mathcal{C}$ of rank $r>0$ and relative degree $r(g-1-d)$. We will denote by $\mathcal{E}$ the *canonical polarization* on $\mathcal{C}$:

$$\mathcal{E}=\begin{cases}\omega_f^{\otimes(g-1-d)}\oplus\mathcal{O}_{\mathcal{C}}^{\oplus(2g-3)}, & d\neq g-1,\\ \mathcal{O}_{\mathcal{C}}, & d=g-1,\end{cases} \tag{3.1}$$

where ω_f is the relative dualizing sheaf of the family f.

Let I be a simple, torsion free, rank-1 sheaf of degree d on a curve C. Then I is *semistable* with respect to a polarization E of rank r, if for every non-empty, proper subcurve $Z\subsetneq C$,

$$\chi(I_Z)\geqslant\frac{-\deg E|_Z}{r}, \tag{3.2}$$

where I_Z is the maximum torsion-free quotient of $I|_Z$. Furthermore, I is *stable* if (3.2) is strict for every Z. Let W (resp. p) be a component of C (resp. a non-singular point of C). Then I is W-*quasistable* (resp. p-quasistable) with respect to a polarization E if I is semistable with respect to E and (3.2) is strict for every Z such that $W\subseteq Z$ (resp. for every Z such that $p\in Z$).

Fix a section $\sigma: B\to\mathcal{C}$ of f through the B-smooth locus of f. A simple, torsion free, rank-1 sheaf $\mathcal{I}$ on $\mathcal{C}\times_B T/T$ is *semistable* (resp. *stable*, resp. σ-*quasistable*) *with respect to a polarization* $\mathcal{E}$, if $\mathcal{I}|_{C_b}$ is semistable (resp. stable, resp. $\sigma(b)$-quasistable) with respect to $\mathcal{E}|_{C_b}$, for every $b\in B$. Consider the subspace $J_{\mathcal{E}}^{\sigma}$ of J_d parametrizing sheaves σ-semistable with respect to the canonical polarization $\mathcal{E}$ defined in (3.1). By [E, Theorem A], $J_{\mathcal{E}}^{\sigma}$ is proper over B. Notice that $J_{\mathcal{E}}^{\sigma}$ finely represents the subfunctor $\mathbf{J}_f^{\sigma}$ of $\mathbf{J}_d$ of the sheaves which are σ-quasistable with respect to $\mathcal{E}$.

Lemma 3.1. *Let C be a stable curve of genus $g\geqslant 3$ and let M be a line bundle on C of degree d. Then*:

(i) *M is semistable (resp. stable) with respect to the canonical polarization if and only if for every non-empty, proper subcurve $Z\subsetneq C$*:

$$\left|\deg M|_Z-\frac{d}{2g-2}\deg\omega_C|_Z\right|\leqslant\frac{\#(Z\cap Z^c)}{2} \tag{3.3}$$

(*resp. the strict inequality holds in* (3.3)).

(ii) *M is W-quasistable with respect to the canonical polarization E if and only if* (3.3) *is satisfied and*:

$$\deg M|_Z-\frac{d}{2g-2}\deg\omega_C|_Z>-\frac{\#(Z\cap Z^c)}{2},$$

for every non-empty, proper subcurve $Z\subsetneq C$ such that $W\subseteq Z$.

Proof. Since $M\in Pic(C)$, M is semistable (resp. stable) with respect to the canonical polarization if and only if for each non-empty, proper subcurve $Z\subsetneq C$:

$$\chi(M|_Z)\geqslant(-\deg E|_Z)/\operatorname{rank}(E) \tag{3.4}$$

(resp. (3.4) is strict for each Z). We have $\chi(M|_Z)=\deg(M|_Z)+1-g_Z$ and $\deg E|_Z=(g-1-d)\deg\omega_C|_Z$. Thus M is semistable (resp. stable) if and only if for each non-empty, proper subcurve $Z\subsetneq C$:

$$\begin{aligned}\deg(M|_Z) &\geqslant g_Z - 1 - \frac{(g-1-d)\deg\omega_C|_Z}{2g-2}\\ &= \frac{d(\deg\omega_C|_Z)}{2g-2} + g_Z - 1 - \frac{\deg\omega_C|_Z}{2} = \frac{d(\deg\omega_C|_Z)}{2g-2} - \frac{\#(Z\cap Z^c)}{2} \end{aligned}\tag{3.5}$$

(resp. if and only if (3.5) is strict for each Z). If M is semistable (resp. stable), we can apply the inequality (3.5) to Z^c, and we get:

$$\deg(M|_Z) \leqslant \frac{d(\deg\omega_C|_Z)}{2g-2} + \frac{\#(Z\cap Z^c)}{2} \tag{3.6}$$

(resp. we get that (3.6) is strict). Then M is semistable (resp. stable) if and only (3.3) holds (resp. the strict inequality holds in (3.3)), for each non-empty, proper subcurve $Z \subsetneq C$. The item (ii) is similar. □

3.2. Admissible twisters

Let $f : \mathcal{C} \to B$ be a smoothing of a semistable curves C. Recall that an *f-twister* of C, or simply a *twister* of C, is a line bundle T on C such that $T \simeq \mathcal{O}_{\mathcal{C}}(D)|_C$, where D is a Cartier divisor of $\mathcal{C}$ with support contained in C.

Definition 3.2. Let C be a stable curve and T a twister of C. We say that a line bundle $M \in \operatorname{Pic} C$ is a *T-spin curve* if $M^{\otimes 2} \simeq \omega_C \otimes T$. If C_0 is an irreducible component of C, a twister T of C is *admissible with respect to* C_0 if the set of T-spin curves is non-empty and every T-spin curve is C_0-quasistable with respect to the canonical polarization $\mathcal{O}_C$.

Recall that T-spin curves have been used in [P] to study degenerations of theta characteristics to non-stable curves.

Lemma 3.3. *Let $f : \mathcal{C} \to B$ be a general smoothing of a stable curve C, where B is the spectrum of a discrete valuation ring, and let T be an f-twister of C. Then the following properties are equivalent*:

(i) *T is admissible with respect to C_0.*
(ii) *The set of T-spin curves is non-empty and there is an integer $r_T \geqslant 0$ and a unique partition of C into non-empty subcurves $Z_0, \dots, Z_{r_T}$ such that*:
   (a) *for every connected component Z'_h of Z_h we have $Z'_h \cap Z_{h-1} \neq \emptyset$, for every $h = 1, \dots, r_T$;*
   (b) *$C_0 \subset Z_0$ and Z_0 is connected*;
   (c) *$T \simeq \mathcal{O}_{\mathcal{C}}(D)|_C$, where $D = \sum_{i=1}^{r_T} i \cdot Z_i$ and*:

$$T \otimes \mathcal{O}_{Z_h} \simeq \begin{cases} \mathcal{O}_{Z_h}(\sum_{p\in Z_h\cap Z_{h+1}} p) & \text{if } h = 0,\\ \mathcal{O}_{Z_h}(\sum_{\substack{p\in Z_h\cap Z_{h+1}\\ q\in Z_{h-1}\cap Z_h}} (p-q)) & \text{if } h = 1,\dots,r_T-1,\\ \mathcal{O}_{Z_h}(-\sum_{p\in Z_{h-1}\cap Z_h} p) & \text{if } h = r_T. \end{cases}\tag{3.7}$$

Proof. Assume that (ii) holds and let L be a T-spin curve. For every non-empty subcurve $Z \subsetneq C$ we have $\deg_Z T = \sum_{p\in Z\cap Z^c} m_p$, for some $m_p \in \{-1,0,1\}$ and hence $|\deg_Z T| \leqslant \#(Z\cap Z^c)$. Thus L is semistable by Lemma 3.1. Let Z be a non-empty, proper subcurve $Z \subsetneq C$. Set $T|_Z \simeq \mathcal{O}_Z(\sum_{p\in Z\cap Z^c} m_p p)$, for $m_p \in \mathbb{Z}$. Then $\deg T|_Z \geqslant -\#(Z\cap Z^c)$. We need to prove that, if $C_0 \subseteq Z$, then $\deg T|_Z > -\#(Z\cap Z^c)$. Assume by contradiction that $\deg T|_Z = -\#(Z\cap Z^c)$. Since $C_0 \subseteq Z_0$, we have $Z \cap Z_0 \neq \emptyset$. We claim that $Z_0 \subseteq Z$. In fact, assume that $Z_0 \not\subseteq Z$. Since Z_0 is connected, there is an irreducible component Y_0 such that $Y_0 \subseteq \overline{Z_0 - Z}$ and $Y_0 \cap Z \neq \emptyset$. Consider a point $p_0 \in Y_0 \cap Z$.

We have $p_0 \in Z \cap Z^c$ and $m_{p_0} = 0$, then $\deg T|_Z > -\#(Z \cap Z^c)$, which is a contradiction. Hence $Z_0 \subseteq Z$. If $Z_0 \nsubseteq Z_1$, then either we have an irreducible component Y_1 such that $Y_1 \subseteq \overline{Z_1 - Z}$ and $Y_1 \cap Z \neq \emptyset$, or there exists a connected component W_1 of Z_1 such that $Z \subseteq W_1^c$. In the first case, arguing as before for Y_0, we get a contradiction. In the second case, the condition (a) implies that $\emptyset \neq W_1 \cap Z_0 \subseteq W_1 \cap Z$. Consider $p_1 \in W_1 \cap Z$. Then by construction $p_1 \in Z \cap Z^c$ and $m_{p_1} = 1$, and hence $\deg T|_Z > -\#(Z \cap Z^c)$, which is a contradiction. Then $Z_1 \subseteq Z$. Iterating, we get an integer $h_Z \in \{0, \dots, r_T - 1\}$ such that $\bigcup_{h=0}^{h_Z} Z_h$ is a connected component of Z. But $m_p = 1$ for each $p \in Z_{h_Z} \cap Z_{h_Z}^c$, thus $\deg T|_{Z_h} = \#(Z_h \cap Z_h^c)$ and hence $\deg T|_Z > \#(Z \cap Z^c)$, which is a contradiction.

Assume that (i) holds. There is a semistable T-spin curve and by Lemma 3.1, for each non-empty subcurve $Z \subsetneq C$, we have $|\deg_Z T| \leqslant \#(Z \cap Z^c)$. Let $C_1 \dots C_\gamma$ be the irreducible components of C. Let $T \simeq \mathcal{O}_{\mathcal{C}}(D)|_C$ for a divisor $D = \sum_{1 \leqslant i \leqslant \gamma} a_i C_i$, $a_i \in \mathbb{Z}$. Since $\mathcal{O}_{\mathcal{C}}(nC)|_C \simeq \mathcal{O}_C$ for every $n \in \mathbb{Z}$, we can assume without loss of generality that $\min_{1 \leqslant i \leqslant \gamma} a_i = 0$. Set:

$$Z_h := \bigcup_{a_i = h} C_i \quad \text{for every } h = 0, \dots, r_T,$$

where $r_T := \max_{1 \leqslant i \leqslant \gamma} a_i$. We prove that r_T and $Z_0, \dots, Z_{r_T}$ satisfy (ii). If $Z_0 = C$ we are done. Otherwise:

$$T \otimes \mathcal{O}_{Z_0} \simeq \mathcal{O}_{Z_0}\Bigg(\sum_{p \in Z_0 \cap Z_0^c} m_p p\Bigg),$$

for some $0 < m_p \in \mathbb{Z}$. Now, $|\deg_{Z_0} T| \leqslant \#(Z_0 \cap Z_0^c)$, hence:

$$\#\big(Z_0 \cap Z_0^c\big) \geqslant |\deg_{Z_0} T| = \sum_{p \in Z_0 \cap Z_0^c} m_p \geqslant \#\big(Z_0 \cap Z_0^c\big).$$

Thus $m_p = 1$, for every $p \in Z_0 \cap Z_0^c$. In particular, we have:

$$T \otimes \mathcal{O}_{Z_0^c} \simeq \mathcal{O}_{Z_0^c}\Bigg(-\sum_{p \in Z_0 \cap Z_0^c} p\Bigg)$$

and hence $Z_1 \supset \{C_i\colon\ C_i \cap Z_0 \neq \emptyset,\ \ C_i \subset Z_0^c\}$. Notice that $Z_1 \neq \emptyset$ and $Z_0 \cap Z_h$ if and only if $h = 1$. Assume that $C \neq Z_0 \cup Z_1$. For every $p \in (Z_1 \cap Z_1^c) - Z_0$, there exists $0 < m_p \in \mathbb{Z}$ such that:

$$T \otimes \mathcal{O}_{Z_1} \simeq \mathcal{O}_{Z_1}\Bigg(\sum_{p \in (Z_1 \cap Z_1^c) - Z_0} m_p p - \sum_{q \in Z_0 \cap Z_1} q\Bigg).$$

Arguing as before for the subcurve $Z_0 \cup Z_1$, we get $m_p = 1$, for $p \in (Z_1 \cap Z_1^c) - Z_0$. Hence:

$$Z_2 \supset \big\{C_i\colon\ C_i \cap Z_1 \neq \emptyset,\ C_i \subset (Z_0 \cup Z_1)^c\big\}.$$

Then $Z_2 \neq \emptyset$ and $Z_1 \cap Z_h \neq \emptyset$ if and only if $|h-1| \leqslant 1$. Iterating, we get that $Z_h \neq \emptyset$ for $h = 0, \dots, r_T$ and $Z_{h_1} \cap Z_{h_2} \neq \emptyset$ if and only if $|h_1 - h_2| \leqslant 1$. Notice that (c) and (3.7) follow by construction.

We show (b). If T is trivial, we have nothing to prove. By (3.7) we have $\deg T|_{Z_0^c} = -\#(Z_0^c \cap Z_0)$. If $C_0 \nsubseteq Z_0$, we get a contradiction, being $C_0 \subseteq Z_0^c$ and T admissible. Then $C_0 \subseteq Z_0$. Assume that Z_0 is not connected and let Z_0' be a connected component of Z_0 such that $C_0 \nsubseteq Z_0'$. Then $C_0 \subseteq (Z_0')^c$ and $\deg T|_{(Z_0')^c} = -\#((Z_0')^c \cap Z_0')$, again a contradiction.

We show (a). Assume that there exists a connected component Z'_h of Z_h such that $Z'_h \cap Z_{h-1} = \emptyset$, for some $h = 1, \dots, r_T$. Then $C_0 \subseteq (Z'_h)^c$ and $\deg T|_{(Z'_h)^c} = -\#((Z'_h)^c \cap Z'_h)$, a contradiction.

Notice that the partition $Z_0, \dots, Z_h$ of C is the unique satisfying (ii). $\square$

Definition 3.4. Keep the notations of Lemma 3.3. We call the partition $Z_0, \dots, Z_{r_T}$ of C *the partition of C induced by T*. We denote by $\mathrm{Ad}_f(C_0)$ the set of the admissible f-twisters T of C with respect to C_0. We say that a node p of C is *T-twisted* if $p \in Z_{i-1} \cap Z_i$, for some $i = 1, \dots, r_T$.

Remark 3.5. Let T and $\overline{T}$ be two admissible f-twisters of C with respect to C_0 and let $Z_0, \dots, Z_{r_T}$ and $\overline{Z}_0, \dots, \overline{Z}_{r_{\overline{T}}}$ be the partitions of C induced respectively by T and $\overline{T}$. Let S (resp. $\overline{S}$) be the set of T-twisted nodes (resp. $\overline{T}$-twisted nodes). If $S = \overline{S}$ and $Z_0 = \overline{Z}_0$, then $T = \overline{T}$. Indeed, the connected components of the two partitions are the same, because they are obtained by taking the desingularization of C at the nodes of S. Since $Z_0 = \overline{Z}_0$, we have $T = \overline{T}$ by condition (a) of Lemma 3.3.

Definition 3.6. Let $f : \mathcal{C} \to B$ be a general smoothing of a nodal curve C. Let X be obtained by blowing-up C at a set Δ of nodes of C. Let $\pi : X \to C$ be the blow-up morphism. Consider the smoothing $f' : \mathcal{X} \to B'$ of X, where B' is the degree-2 covering of B, totally ramified over $0 \in B$, and $\mathcal{X}$ is the blow-up of $\mathcal{C} \times_{B'} B$ at Δ. Fix $M \in \mathrm{Pic}\, C$ and $L \in \mathrm{Pic}\, X$. We say that L and M are *f-related* if there exists an f'-twister T of X such that $L \simeq \pi^* M \otimes T$.

Lemma 3.7. *Let $f : \mathcal{C} \to B$ be a general smoothing of a stable curve C of genus $g \geqslant 3$, where B is the spectrum of a discrete valuation ring. Let T be an admissible f-twister of C. Let $Z_0, \dots, Z_{r_T}$ be the partition of C induced by T. Assume that M is a T-spin curve and that a representative (X, L, α) of some $\xi \in \overline{S_C}(\omega_C)$ fullfills the following properties*:

(i) *X is obtained by blowing-up C at the T-twisted nodes*;
(ii) *for every $h = 0, \dots, r_T$, the restriction of L to Z_h is*:

$$L \otimes \mathcal{O}_{Z_h} \simeq \begin{cases} M \otimes \mathcal{O}_{Z_h}(-\sum_{p \in Z_h \cap Z_{h+1}} p) & \text{if } h = 0, \dots, r_T - 1, \\ M \otimes \mathcal{O}_{Z_h} & \text{if } h = r_T. \end{cases}$$

Then there exists a representative (X, L_M, α_M) of ξ such that M and L_M are f-related.

Proof. Let $B' \to B$ be the degree 2 cover of B, totally ramified over $0 \in B$. Let $\mathcal{X}$ be obtained by blowing-up $\mathcal{C} \times_B B'$ at the set of the T-twisted nodes of C. Then the projection $f' : \mathcal{X} \to B'$ is a smoothing of the fiber $X = (f')^{-1}(0)$ and X is obtained by blowing-up C at the set of T-twisted nodes. Let $\pi : X \to C$ be the induced blow-up morphism. Notice that $\widetilde{X}$, the residual in X of the union of the exceptional component of X, is the disjoint union of $Z_0, \dots, Z_{r_T}$. Furthermore, $\mathcal{X}$ is smooth at every node lying on an exceptional component of X and has a singularity of type A_1 at the remaining nodes. Let $\mathcal{E}_h$ be the set of exceptional components of X intersecting Z_{h-1} and Z_h, for each $h = 1, \dots, r_T$. Consider the Cartier divisor of $\mathcal{X}$:

$$D_M = -\sum_{h=1}^{r_T} \left(h \cdot Z_h + h \cdot \sum_{E \in \mathcal{E}_h} E \right).$$

Pick the f'-twister $T_M = \mathcal{O}_{\mathcal{X}}(D_M) \otimes \mathcal{O}_X$ of X. Set $L_M := \pi^* M \otimes T_M \in \mathrm{Pic}\, X$. By construction, L_M and M are f-related. We are done if we show that we can construct a representative (X, L_M, α_M) of ξ. First of all, we define α_M as follows. By construction, $L_M|_E \simeq \mathcal{O}_E(1)$ for every exceptional component E and by condition (ii) we get $L_M|_{Z_h} = L|_{Z_h}$ for every $h = 0, \dots, r_T$. By definition, $M^{\otimes 2} \simeq \omega_C \otimes T$

and, by formula (3.7) of Lemma 3.3:

$$(\omega_C \otimes T)|_{Z_h} \simeq \begin{cases} \omega_{Z_h}(\sum_{p\in Z_h\cap Z_{h+1}} 2p) & \text{if } h=0,\dots,r_T-1, \\ \omega_{Z_h} & \text{if } h=r_T. \end{cases}$$

Thus, $(L_M|_{Z_h})^{\otimes 2} \simeq \omega_{Z_h}$, for every $h=0,\dots,r_T$. Let $\alpha_M:(L_M)^{\otimes 2} \to \pi^*(\omega_X)$ be the homomorphism which agrees on each Z_h with:

$$\alpha_h : (L_M|_{Z_h})^{\otimes 2} \simeq \omega_{Z_h} \simeq \pi^*(\omega_C) \otimes \mathcal{O}_{Z_h}\Bigg(-\sum_{p\in Z_h\cap Z_h^c} p\Bigg) \hookrightarrow \pi^*(\omega_C) \otimes \mathcal{O}_{Z_h}$$

and which is zero on the exceptional components of X. Now, $\widetilde{X} = \bigcup_{i=0}^{r_T} Z_i$, then $L_M|_{\widetilde{X}} \simeq L|_{\widetilde{X}}$ and (X, L_M, α_M) is a representative of ξ by Remark 2.1. $\square$

Let $f:\mathcal{C}\to B$ be a general smoothing of a stable curve C. For any f-twister T of $\mathcal{C}$ consider the moduli space:

$$\overline{S_f}(\omega_f \otimes T) \to B$$

whose fiber over $b\in B$ parametrizes limit square roots of $\omega_f \otimes T \otimes \mathcal{O}_{C_b}$. These moduli spaces are isomorphic away from the special fiber. Hence they have the same normalization, which, in the notations of Theorem 2.2, we write as:

$$\nu_T : \overline{S_f^{\nu}}(\omega_f) \to \overline{S_f}(\omega_f \otimes T).$$

Let $S_f(\omega_f \otimes T)$ be the open subscheme of $\overline{S_f}(\omega_f \otimes T)$ parametrizing limit square root supported on stable curves. Notice that $S_f(\omega_f \otimes T)$ is étale over B, by [CCC, 4.1]. In particular, there is an immersion $S_f(\omega_f \otimes T) \hookrightarrow \overline{S_f^{\nu}}(\omega_f)$.

Remark 3.8. Let $f:\mathcal{C}\to B$ be a smoothing of a nodal curve C and let $\mathcal{G}\in \operatorname{Pic}(\mathcal{C})$. Let $L\in \operatorname{Pic}(C)$ be endowed with an isomorphism $\iota_0 : L^{\otimes 2} \to \mathcal{G}|_C$. By [CCC, Remark 3.0.6.], up to shrinking B to a complex neighborhood of 0, there exists a line bundle $\mathcal{L}\in \operatorname{Pic}(\mathcal{C})$ extending L and an isomorphism $\iota : \mathcal{L}^{\otimes 2} \to \mathcal{G}$ extending ι_0. Moreover, if $(\mathcal{L}', i')$ is another extension of (L, ι_0), then there is an isomorphism $\chi : \mathcal{L} \to \mathcal{L}'$, restricting to the identity, and with $\iota = \iota' \circ \chi^{\otimes 2}$.

Theorem 3.9. *Let $f:\mathcal{C}\to B$ be a general smoothing of a stable curve C of genus $g \geqslant 3$ with $\operatorname{Aut}(C)=\{id\}$. Let C_0 be an irreducible component of C. Then*:

$$N\big(S\big(\omega_f^*\big)\big) \simeq \frac{\bigcup_{T\in \operatorname{Ad}_f(C_0)} S_f(\omega_f \otimes T)}{\sim},$$

where $\sim$ denotes the gluing along the generic fiber of $S_f(\omega_f \otimes T) \to B$.

Assume that f admits enough sections through the B-smooth locus of f and let σ be a section of f through the B-smooth locus of $\mathcal{C}$ such that $\sigma(0)\in C_0$. Fix the canonical polarization $\mathcal{E} = \mathcal{O}_{\mathcal{C}}$ on $\mathcal{C}$. If $(J_{\mathcal{E}}^{\sigma})^{free}$ is the open subscheme of $J_{\mathcal{E}}^{\sigma}$ parametrizing locally free sheaves, then there exists an immersion:

$$\psi_f : N\big(S\big(\omega_f^*\big)\big) \hookrightarrow \big(J_{\mathcal{E}}^{\sigma}\big)^{free}.$$

Proof. We can assume without loss of generality that B is the spectrum of a discrete valuation ring. Recall that $S_f(\omega_f \otimes T) \hookrightarrow \overline{S_f^{\nu}}(\omega_f)$, for every twister T. By Theorem 2.2, it suffices to show the equivalence of the following properties, for every $\xi \in \overline{S_f}(\omega_f)$:

(i) $\overline{S_f^{\nu}}(\omega_f) \to B$ is étale at $\xi' \in \nu^{-1}(\xi)$;
(ii) there exists a unique $T \in \mathrm{Ad}_f(C_0)$ such that $\overline{S_f^{\nu}}(\omega_f)$ and $S_f(\omega_f \otimes T)$ are isomorphic, locally at $\xi' \in \nu^{-1}(\xi)$.

We show (i) $\Rightarrow$ (ii). Let (X, L, α) be any representative of ξ. We show that there exists a T satisfying (ii). If $X = C$, then it suffices to set $T = \mathcal{O}_C \in \mathrm{Ad}_f(C_0)$. Assume that $X \neq C$ and let $\pi : X \to C$ be the blow-up map and $\widetilde{X}_0, \ldots, \widetilde{X}_c$ be the connected components of $\widetilde{X}$, corresponding to the vertices of Σ_X. Let v_i be the vertex of Σ_X corresponding to $\widetilde{X}_i$. By Theorem 2.2, the graph Σ_X is bipartite. Assume that $C_0 \subset \widetilde{X}_0$ and set $A_0 := \{v_0\}$. For every $i \geqslant 1$, define inductively the set A_i as the set of vertices v of Σ_X such that there exists an edge containing v and a vertex of A_{i-1}. Let $A_0, \ldots, A_r$ be the non-empty sets defined in this way. Abusing notation, we can see $\widetilde{X}_i$ as a subcurve of C. Consider the divisor $D = \sum_{i=0}^{r} \sum_{v_j \in A_i} i \cdot \widetilde{X}_j$ of $\mathcal{C}$. Set $T := \mathcal{O}_{\mathcal{C}}(D)|_C$. Notice that $Z_0 = \widetilde{X}_0$ and T satisfies the conditions of Lemma 3.3 (ii), then $T \in \mathrm{Ad}_f(C_0)$. Let $Z_0, \ldots, Z_{r_T}$ be the partition of C induced by T. Being Σ_X bipartite, each edge of Σ_X has a vertex in A_{i-1} and the other vertex in A_i, for some $i = 1, \ldots, r$. In particular, X is obtained by blowing-up C at the T-twisted nodes. Consider the subset $\mathcal{M}$ of $S_f(\omega_f \otimes T)$ defined as the set of T-spin curve $M \in \mathrm{Pic}\, C$ satisfying for every $h = 0, \ldots, r_T$:

$$M \otimes \mathcal{O}_{Z_h} \simeq \begin{cases} L \otimes \mathcal{O}_{Z_h}\big(\sum_{p \in Z_h \cap Z_{h+1}} p\big) & \text{if } h = 0, \ldots, r_T - 1, \\ L \otimes \mathcal{O}_{Z_h} & \text{if } h = r_T. \end{cases}$$

Notice that $S_f(\omega_f \otimes T) \to B$ is étale at each $M \in \mathcal{M}$. Then $S_f(\omega_f \otimes T)$ and $\overline{S_f^{\nu}}(\omega_f)$ are isomorphic, locally at each $M \in \mathcal{M}$. Our goal is to show that $\overline{S_f^{\nu}}(\omega_f)$ and $S_f(\omega_f \otimes T)$ are isomorphic, locally at every $\xi' \in \nu^{-1}(\xi)$. It is enough to show that $\mathcal{M} = \nu^{-1}(\xi)$. By Lemma 3.7, for every $M \in \mathcal{M}$, there is a representative (X, L_M, α_M) of ξ such that L_M and M are f-related. Keep the notations of Definition 3.6. Since L_M and M are f-related, it follows from Remark 3.8 that L_M and $\pi^* M$ are limits of the same family of theta characteristics on the family $f'|_{\mathcal{X} - X}$. Thus, $M \in \nu^{-1}(\xi)$ and hence $\mathcal{M} \subset \nu^{-1}(\xi)$. Now, the ramification index of $\psi : \overline{S_f}(\omega_f) \to B$ at ξ is $2^{b_1(\Sigma_X)}$, then $|\nu^{-1}(\xi)| \leqslant 2^{b_1(\Sigma_X)}$ and, by construction, $|\mathcal{M}| = 2^{b_1(\Sigma_X)}$. This implies $\mathcal{M} = \nu^{-1}(\xi)$.

We claim that T is uniquely determined within $\mathrm{Ad}_f(C_0)$, i.e. if $\overline{S_f^{\nu}}(\omega_f)$ and $S_f(\omega_f \otimes \overline{T})$ are isomorphic, locally at $\xi' \in \nu^{-1}(\xi)$ for some $\overline{T} \in \mathrm{Ad}_f(C_0)$, then $\overline{T} = T$. Indeed, in this case, there exists a $\overline{T}$-spin curve $\overline{M}$ such that $\overline{M} \in \nu^{-1}(\xi)$. We claim that X is obtained by blowing-up C at the $\overline{T}$-twisted nodes. Otherwise, let $\overline{X}$ be obtained by blowing-up C at the $\overline{T}$-twisted nodes, with $\overline{X} \neq X$. By Lemma 3.7, there exists $\bar{\xi} \in \overline{S}_C(\omega_C)$, with a representative $(\overline{X}, \overline{L}, \overline{\alpha})$, where $\overline{L}$ is f-related to $\overline{M}$. Arguing as before, we get $\overline{M} \in \nu^{-1}(\bar{\xi})$ and hence $\xi = \bar{\xi}$, contradicting Remark 2.1. Now, let $\overline{Z}_0, \ldots, \overline{Z}_{r_{\overline{T}}}$ be the partition of C induced by $\overline{T}$. Since X is obtained by blowing-up C at the $\overline{T}$-twisted nodes, the set of T-twisted nodes coincides with the set of $\overline{T}$-twisted nodes. Then $Z_0 \cap Z_0^c$ are $\overline{T}$-twisted nodes. Being $\emptyset \neq C_0 \subseteq Z_0 \cap \overline{Z}_0$ and $\overline{Z}_0$ connected, we have $\overline{Z}_0 \subseteq Z_0$. Arguing similarly we get $Z_0 \subseteq \overline{Z}_0$ and hence $\overline{Z}_0 = Z_0$. Then $\overline{T} = T$, by Remark 3.5. The implication (ii) $\Rightarrow$ (i) is trivial.

Now we prove the second part. First of all, we show the existence of a morphism $S_f(\omega_f \otimes T) \to J^{\sigma}_{\mathcal{E}}$, for every $T \in \mathrm{Ad}_f(C_0)$. In fact, let $\boldsymbol{\mathcal{S}}_f(\omega_f \otimes T)$ be the subfunctor of the functor $\overline{\boldsymbol{\mathcal{S}}_f}(\omega_f \otimes T)$ defined in (2.1), associating to a locally Noetherian B-scheme T the set of isomorphism classes of limit square roots of $\omega_{f'}$ supported on $\mathcal{C} \times_B T$, for $f' : \mathcal{C} \times_B T \to T$ the first projection. By definition of admissible twister, we have a transformation of functors:

$$\boldsymbol{\mathcal{S}}_f(\omega_f \otimes T) \to \mathbf{J}^{\sigma}_{\mathcal{E}} \xrightarrow{\sim} \mathbf{Hom}\big(-, J^{\sigma}_{\mathcal{E}}\big).$$

Now, $S_f(\omega_f \otimes T)$ coarsely represents $\mathcal{S}_f(\omega_f \otimes T)$. Therefore, we get a morphism $S_f(\omega_f \otimes T) \to J^{\sigma}_{\mathcal{E}}$. By the first part of the theorem, we have:

$$N\big(S\big(\omega_f^*\big)\big) \simeq \frac{\bigcup_{T \in \mathrm{Ad}_f(C_0)} S_f(\omega_f \otimes T)}{\sim}$$

hence we get a morphism $\psi_f : N(S(\omega_f^*)) \to J^{\sigma}_{\mathcal{E}}$, which is injective because the line bundles parametrized by the points of $N(S(\omega_f^*))$ over $0 \in B$ are non-isomorphic T-spin curves. Now, $\psi_f : N(S(\omega_f^*)) \to \mathrm{Im}\,\psi_f$ is an injective B-morphism and $N(S(\omega_f^*))$ is B-smooth. Then $\mathrm{Im}\,\psi_f$ is B-smooth and ψ_f is an immersion. By construction $\mathrm{Im}\,\psi_f \subset (J^{\sigma}_{\mathcal{E}})^{free}$. $\square$

Acknowledgments

I wish to thank L. Caporaso for fundamental suggestions. I thank S. Busonero, E. Esteves and F. Noseda for useful discussions.

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